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Appendix 1
Derivations

from

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PROOF OF THE ITERATIVE DECONVOLUTION

A detailed discussion of the iterative deconvolution has been presented in Ioup and Ioup (1983). Here only the part of the derivation which is necessary for LOTEM is shown.

In the field, the signal y(t) and the system response function s(t) are measured. The wanted real transient is then x(t), where y(t) is a convolution of x(t) with the system response functions s(t).

$$y(t) = s(t) * x(t) = \int_{-\infty}^{\infty} s(u) \cdot x(t-u)\, du \tag{A.1.1}$$

Following the convolution theorem (Bracewell, 1965) in the transform domain s for example yields

$$Y(s) = S(s) \cdot X(s) \tag{A.1.2}$$

For the iterative deconvolution one now can define a van Citert–iteration.

$$a_0 = y(t) \tag{A.1.3}$$

$$\begin{aligned} a_1 &= a_0 + \left[y(t) - a_0 * s(f) \right] \\ &= y(t) + \left[y(t) - y(t) * s(t) \right] \end{aligned} \tag{A.1.4}$$

$$\begin{aligned} a_2 &= a_1 + \left[y(t) - a_1 * s(t) \right] \\ &\bullet \\ &\bullet \\ &\bullet \\ a_m &= a_{m-1} + \left[y(t) - a_{m-1} * s(t) \right] \end{aligned} \tag{A.1.5}$$

If you substitute equation (1.2) into equations (1.3) – (1.5), you get in the transform domain:

$$A_0(s) = Y(s) \tag{A.1.6}$$

$$\begin{aligned} A_1(s) &= Y(s) + \left[Y(s) - Y(s)\ S(s) \right] \\ &= Y(s)\left[1 + (1 - S(s)) \right] \end{aligned} \tag{A.1.7}$$

$$\begin{aligned} A_2(s) &= A_1(s) + \left[Y(s) - A_1(s)\, S(s) \right] \\ &= Y(s)\left[1 + 1 - S(s) + 1 - S(s) - S(s) + S(s)^2 \right] \\ &= Y(s)\left[1 + (1 - S(s)) + (1 - S(s))^2 \right] \\ &\vdots \\ A_m(s) &= A_{m-1}(s) + \left[Y(s) - A_{m-1}(s)\ S(s) \right] \\ &= Y(s)\left[1 + (1 - S(s)) + (1 - S(s))^2 + \ldots + (1 - S(s))^m \right] \\ &= Y(s) \sum_{i=0}^{m} (1 - S(s))^i \end{aligned} \tag{A.1.8}$$

The series (1.8) converges for $(1 - S(s)) < 1$. This is always the case for transient electromagnetics, because the transients are causal and the system response function is normalized to 1. As limit of equation (1.8) we obtain:

$$\begin{aligned} \lim_{m \to \infty} A_m &= \lim_{m \to \infty} Y(s) \sum_{i=0}^{m} (1 - S(s))^i \\ &= \frac{Y(s)}{1 - (1 - S(s))} = \frac{Y(s)}{S(s)} = X(s) \end{aligned} \tag{A.1.9}$$

It was proven that the van Citert iteration (1.6) to (1.8) in the transform domain leads to the wanted quantity X (s). That means that we will get x (t) in the time domain and by doing so one can carry out the deconvolution.

SINGULAR VALUE DECOMPOSITION

The singular value decomposition has helped us to obtain a more objective estimate of the inversion parameters. It is well described in Lancos (1958) and Jackson (1972). Here we include a summary of the SVD for completeness and since it is one of the most frequently requested background details.

The goal is the solution of the linear system:

$$(\mathbf{J}^T \mathbf{W}^2 \mathbf{J} + \mathrm{K}^2 \mathbf{I})\,\Delta = \mathbf{J}^T \mathbf{W}^2 \mathbf{g} \tag{A.2.1}$$

where $\mathbf{J}$ is the Jacobian, $\mathbf{W}$ the weighing matrix, $\mathbf{I}$ the identity matrix, K the damping factor, Δ the parameter difference vector and $\mathbf{g}$ the discrepancy vector (see chapter 4). For simplicity reason the weighing matrix may be omitted.

This goal can be reached by spectrally decomposing the Jacobian. After Lancos (1958) a spectral decomposition exists for every n x m matrix $\mathbf{J}$ with rang $(\mathbf{J}) = p \leq \min(m,n)$ such that

$$\mathbf{J} = \mathbf{U}\,\mathbf{S}\,\mathbf{V}^T \tag{A.2.2}$$

$\mathbf{S}$ is a m x m diagonal matrix. Its elements contain the non negative roots of the eigenvalues of $\mathbf{J}^T\mathbf{J}$. With the maximum number of linear independent column (or row) vectors rang (J) = p, only p elements of $\mathbf{S}$ are different from zero (Jackson, 1972). The elements of $\mathbf{S}$ are ordered such that:

$$S_1 \geq S_2 \geq S_3 \ldots \geq S_p > 0;\ S_{p+1}, \ldots S_m = 0 \tag{A.2.3}$$

U is a n x m matrix with p orthonormal data space eigenvectors in its columns corresponding to the non-zero eigenvectors.

V is a m x m matrix containing the eigenvectors of the parameter space in its columns.

This means, if V_j is a column vector of $\mathbf{V}$ with $j \leq p$ then

$$\mathbf{J}^T \mathbf{J}\,\mathbf{V}_j = S_j^2\,\mathbf{V}_j$$

if $\mathbf{u}_i$ is a column vector of $\mathbf{U}$ with $i \leq p$ then

$$\mathbf{J}\,\mathbf{J}^T\,\mathbf{U}_i = S_i^2\,\mathbf{U}_i$$

Since the column vectors are orthonormal the matrices $\mathbf{U}$ and $\mathbf{V}$ satisfy

$$\mathbf{U}^T\mathbf{U} = \mathbf{V}^T\mathbf{V} = \mathbf{V}\,\mathbf{V}^T = \mathrm{I}_m \tag{A.2.4}$$

If we define two diagonal matrices $\mathbf{S}^*$ and $\mathbf{T}$ such that

$$S^*_{ii} = \begin{cases} \dfrac{1}{S_{ii}} & \text{for } S_{ii} > 0 \\ 0 & \text{otherwise} \end{cases} \tag{A.2.5}$$

$$T_{ii} = \frac{S_{ii}^2}{S_{ii}^2 + K^2} \tag{A.2.6}$$

the solution of equation (2.1) (without weighing matrix) is:

$$\Delta = \mathbf{V\,T\,S^*\,U^T}\ g \tag{A.2.7}$$

Proof: Substitute equation (2.7) in equation (2.1) (without weights).

$$\begin{aligned}
(\mathbf{J^T J} + K^2\,\mathbf{I}_m)\,\mathbf{VTS\,U^T\,g} &= (\,\mathbf{VS^T U^T U\,S\,V^T} + K^2\,\mathbf{I}\,)\,\mathbf{VTS^*\,U^T\,g} = \\
&= (\,\mathbf{VS^2\,V^T\,VTS^*\,U^T} + \mathbf{V}\,K^2\,\mathbf{I\,TS^*\,U^T}\,)\,\mathbf{g} = \\
&= (\,\mathbf{V}(\mathbf{S^2\,TS^*} + K^2\,\mathbf{I\,TS^*}\,)\mathbf{U^T}\,)\,\mathbf{g} = \\
&= (\,\mathbf{V}(\mathbf{S^2} + K^2\,\mathbf{I}\,)\,\mathbf{TS^*\,U^T}\,)\,\mathbf{g} = \\
&= \mathbf{VS\,U^T\,g} = \mathbf{I^T\,g}
\end{aligned} \tag{A.2.7}$$

SOLUTION OF MAXWELL'S EQUATIONS USING SCALAR POTENTIALS

In this section we derive step by step the solution to Maxwell's equations which leads to the equations for the electromagnetic fields

We use the quasi-stationary Maxwell's equations:

$$\nabla \times \mathbf{E} = -\dot{\mathbf{B}} = \text{curl E} \tag{A.3.1}$$

$$\nabla \times \mathbf{H} = \mathbf{j} = \text{curl H} \tag{A.3.2}$$

$$\nabla\,\mathbf{D} = 0 \tag{A.3.3}$$

$$\nabla\,\mathbf{B} = 0 \tag{A.3.4}$$

and the material equations:

$$\mathbf{j} = \sigma\,\mathbf{E} \tag{A.3.5}$$

$$\mathbf{B} = \mu\,\mathbf{H} \tag{A.3.6}$$

$$\mathbf{D} = \epsilon\,\mathbf{E} \tag{A.3.7}$$

where **E** is the electric field, **B** is the magnetic induction, **H** is the magnetic field, **j** is the current density and **D** is the electric displacement. σ is the conductivity, $\mu \approx \mu_0 = 4\pi \cdot 10^{-7}$ Vs/Am is the magnetic permeability, and ϵ is the dielectric permittivity. Since **j** and **B** are both divergence–free, they can be decomposed into a toloidal mode (index E) and a poloidal mode (index M), which are described by scalar Debye potentials ϕ_E and ϕ_M:

$$
\begin{aligned}
\mathbf{B} &= \mathbf{B}_E + \mathbf{B}_M = \nabla \times \nabla \times (\mathbf{e}_z\, \phi_E) + \nabla \times (\mathbf{e}_z\, \sigma\, \mu_0\, \phi_M) \\
\mathbf{j} &= \mathbf{j}_E + \mathbf{j}_M = -\nabla \times (\mathbf{e}_z\, \sigma\, \dot{\phi}_E) + \nabla \times (\mathbf{e}_z\, \sigma\, \phi_M)
\end{aligned}
\tag{A.3.8}
$$

where $\mathbf{e}_z$ is the unit vector in z–direction and $\mathbf{j}_E$ is the toroidal current density. It has no vertical component and describes horizontal current loops. $\mathbf{j}_M$, the poloidal density, has no horizontal component and describes vertical current loops.

From equations (3.5) and (3.6) we get:

$$
\begin{aligned}
\mathbf{E}_E &= -\nabla \times (\mathbf{e}_z\, \dot{\phi}_E) & \mathbf{E}_M &= \frac{1}{\sigma}\, \nabla \times \nabla \times (\mathbf{e}_z\, \sigma\, \phi_M) \\
\mathbf{H}_E &= \frac{1}{\mu_0}\, \nabla \times \nabla \times (\mathbf{e}_z\, \phi_E) & \mathbf{H}_M &= \nabla \times (\mathbf{e}_z\, \sigma\, \phi_M)
\end{aligned}
\tag{A.3.9}
$$

Faraday's induction law (3.1) is already satisfied by $\mathbf{E}_E$, while $\mathbf{H}_M$ satisfies Ampere's law (3.2). To satisfy (3.1) and (3.2) also with $\mathbf{E}_M$ and $\mathbf{H}_E$, the two scalar functions ϕ_E and ϕ_M must be solutions of the differential equations

$$
\begin{aligned}
&\nabla^2\, \phi_E = \mu_0\, \sigma\, \dot{\phi}_E \\
&\nabla \left(\frac{1}{\sigma}\, \nabla (\sigma\, \phi_M)\right) = \mu_0\, \sigma\, \dot{\phi}_M
\end{aligned}
\tag{A.3.10}
$$

ϕ_E and ϕ_M are two independent solutions of Maxwell's equations. ϕ_E contains no vertical electric field component and is called the tangential electric polarization or TE mode. ϕ_M has no vertical component of the magnetic field and is called the tangential magnetic polarization or TM mode. The field components of **E** and **H** can be written:

TE mode

$$\mathbf{E}_{Ex} = -\frac{\delta}{\delta y}\, \dot{\phi}_E$$

$$\mathbf{E}_{Ey} = \frac{\delta}{\delta x}\, \dot{\phi}_E$$

$$\mathbf{E}_{Ez} = 0$$

TM mode

$$\mathbf{E}_{Mx} = \frac{1}{\sigma}\, \frac{\delta^2}{\delta x\, \delta z}\, (\sigma\, \phi_M)$$

$$\mathbf{E}_{My} = \frac{1}{\sigma}\, \frac{\delta^2}{\delta y\, \delta z}\, (\sigma\, \phi_M)$$

$$\mathbf{E}_{Mz} = -\left(\frac{\delta^2}{\delta x^2} + \frac{\delta^2}{\delta y^2}\right) \phi_M$$

$$H_{Ex} = \frac{1}{\mu_0} \frac{\delta^2}{\delta x \delta z} \phi_E \qquad H_{Mx} = \sigma \frac{\delta}{\delta y} \phi_M$$

$$H_{Ey} = \frac{1}{\mu_0} \frac{\delta^2}{\delta y \delta z} \phi_E \qquad H_{My} = -\sigma \frac{\delta}{\delta x} \phi_M$$

$$H_{Ez} = -\frac{1}{\mu_0} \left(\frac{\delta^2}{\delta x^2} + \frac{\delta^2}{\delta y^2} \right) \phi_E \qquad H_{Mz} = 0 \qquad \text{(A.3.11)}$$

These fields can be described by closed field lines. In particular, the field lines for $\mathbf{j}_E$, $\mathbf{E}_E$, $\mathbf{B}_M$ and $\mathbf{H}_M$ are in horizontal planes and are described by ϕ_E = constant or ϕ_M = constant.

The equations (3.10) are only valid, if σ is continual. At layer boundaries, where σ is not continuous, one can derive the following constraints for the Debye potentials:

$$\phi_E , \ \frac{\delta \phi_E}{\delta z} , \ \sigma \phi_M \ \text{ and } \ \frac{1}{\sigma} \frac{\delta}{\delta z} (\sigma \phi_M) \ \text{ are continuous.} \qquad \text{(A.3.12)}$$

In free space it can be assumed that $\sigma = 0$, and thus from the third constraint we get that $\phi_M (z=0) = 0$. This is another expression for the fact that there is no vertical current density at $z = 0$, which has the important consequence that

$$\phi_M = 0 \ \text{ in } \ z > 0 \text{ , if all sources are in } z < 0 \text{ and } \sigma = \sigma(z). \qquad \text{(A.3.13)}$$

This means that the induced currents flow in horizontal directions for the arbitrary sources in free space, which are inductively coupled to the layered half-space. On the other hand there are vertical currents, if the coupling of the source is galvanic which is the case for the LOTEM method.

To solve the differential equations (3.10), the scalar potentials ϕ_E and ϕ_M are decomposed into partial waves (Fourier components):

$$\phi_{E,M} (\mathbf{r}, t) = \iiint_{-\infty}^{\infty} f_{E,M} (z, \mathbf{k}, \omega) \, e^{i(kr + \omega t)} d^2 \mathbf{k} \, d\omega \qquad \text{(A.3.14)}$$

where $\mathbf{k} = (k_x, k_y, 0)$ and $d^2 \mathbf{k} = dk_x dk_y$.

Substitution of (3.12) into (3.10) yields:

$$\textbf{TE–Mode:} \qquad \frac{d^2 f_E(z)}{d\, z^2} = \alpha^2(z)\, f_E(z)$$

$$\textbf{TM–Mode:} \qquad \frac{d}{d\, z}\left(\frac{1}{\sigma}\,\frac{d}{d\, z}\,(\sigma\, f_M(z))\right) = \alpha^2(z)\, f_M(z)\ . \tag{A.3.15}$$

where $\alpha^2(z) = k^2 + i\,\omega\,\mu_0\,\sigma\,(z)\ ,\quad k^2 = |\mathbf{k}|^2$

Using $\dfrac{\delta}{\delta\, x} = i\, k_x\ ,\quad \dfrac{\delta}{\delta\, y} = i\, k_y$ and $\dfrac{\delta}{\delta\, t} = i\,\omega\ ,$

we get for the frequency–wavenumber domain from (11):

$$\hat{E}_{Ex} = \omega\, k_y\, f_E \qquad\qquad \hat{E}_{Mx} = i\, k_x \frac{d\, f_M}{d\, z}$$

$$\hat{E}_{Ey} = -\,\omega\, k_y\, f_E \qquad\qquad \hat{E}_{My} = i\, k_y \frac{d\, f_M}{d\, z}$$

$$\hat{E}_{Ez} = 0 \qquad\qquad \hat{E}_{Mz} = k^2\, f_M$$

$$\hat{H}_{Ex} = \frac{i\, k_x}{\mu_0}\,\frac{d\, f_E}{d\, z} \qquad\qquad \hat{H}_{Mx} = i\, k_y\, \sigma\, f_M$$

$$\hat{H}_{Ey} = \frac{i\, k_y}{\mu_0}\,\frac{d\, f_E}{d\, z} \qquad\qquad \hat{H}_{My} = -\, i\, k_x\, \sigma\, f_M$$

$$\hat{H}_{Ez} = \frac{k^2}{\mu_0}\, f_E \qquad\qquad \hat{H}_{Mz} = 0 \tag{A.3.16}$$

where ($\hat{}$) stands for the frequency–wavenumber domain.

For the following it is useful to define the impedances:

$$Z_E(z, \mathbf{k}, \omega) := \frac{\hat{E}_{Ex}(z, \mathbf{k}, \omega)}{\hat{H}_{Ey}(z, \mathbf{k}, \omega)} = -\,\frac{\hat{E}_{Ey}(z, \mathbf{k}, \omega)}{\hat{H}_{Ex}(z, \mathbf{k}, \omega)} = i\,\omega\,\mu_0 \left(\frac{-\, f_E\,(z, \mathbf{k}, \omega)}{\frac{d}{d\, z}\, f_E\,(z, \mathbf{k}, \omega)}\right) \tag{A.3.17}$$

$$Z_M(z, \mathbf{k}, \omega) := \frac{\hat{E}_{Mx}(z, \mathbf{k}, \omega)}{\hat{H}_{My}(z, \mathbf{k}, \omega)} = - \frac{\hat{E}_{My}(z, \mathbf{k}, \omega)}{\hat{H}_{Mx}(z, \mathbf{k}, \omega)} = i\omega\mu_0 \; - \frac{\frac{d}{dz} f_M(z, \mathbf{k}, \omega)}{\sigma f_M(z, \mathbf{k}, \omega)}$$

From the impedances (3.17) we define the reciprocal modified impedances:

$$B_{EM}(z, \mathbf{k}, \omega) = \frac{i\omega\mu_0}{Z_E(z, \mathbf{k}, \omega)} = - \frac{\frac{d}{dz} f_E(z, \mathbf{k}, \omega)}{f_E(z, \mathbf{k}, \omega)}$$

$$B_M(z, \mathbf{k}, \omega) = \sigma Z_M(z, \mathbf{k}, \omega) = - \frac{\frac{d}{dz} f_M(z, \mathbf{k}, \omega)}{f_M(z, \mathbf{k}, \omega)} \tag{A.3.18}$$

In a horizontally layered half-space within the homogeneous isotropic layer m, f_E and f_M must satisfy the equations:

$$\frac{d^2 f_E}{dz^2} \; f_E(z) = \alpha_m^2 f_E(z)$$

$$\frac{d^2 f_M}{dz^2} \; f_M(z) = \alpha_m^2 f_M(z) \tag{A.3.19}$$

$$\text{where } \alpha_m^2 = k^2 + i\omega\mu_0\sigma_m$$

From (3.12) follows that

$$f_E, \; \frac{d f_E}{dz}, \; \sigma f_M \text{ and } \frac{d f_M}{dz}$$

are continuous at the layer boundaries. Moreover, f_E and f_M should be 0 for $z \to \infty$.

The calculation of $f_E(z)$ and $f_M(z)$ will be done in two steps: first a recursive algorithm for the calculation of B_E and B_M at the layer boundaries h_M will be derived. The general solution for equation (3.19) is:

$$f_{E,M}(z) = b^-_{E,M,m} \, e^{-\alpha_m(z-h_m)} + b^+_{E,M,m} \, e^{\alpha_m(z-h_m)}, \text{ for } h_m \le z \le h_{m+1}$$

and (A.3.20)

$$f_{E,M}(z) = b^-_{E,M,n} \, e^{-\alpha_n(z-h_n)}, \text{ for } z \ge h_n .$$

where n is the number of layers and h_n is the last layer boundary. For $z \ge h_n$ only the "–" part exists, because $f_{E,M} = 0$ for $z \to \infty$.

The solution is:

$$f\ (z) = (\ e^{-kz} + r_0\ e^{kz}\ )\ f_0^e\ (\ \mathbf{k},\ \omega)\ ,\quad 0 \geq Z \geq -\epsilon \tag{A.3.24}$$

where the index "e" stands for "external".

The first term describes the source field:

$f_E^e(z,\ \mathbf{k},\ \omega) = f_0^e\ (\ \mathbf{k},\ \omega)\ e^{-kz}$, the second term is in the induced field.

Since f_E and $\dfrac{d f_E}{d z}$ must be continual at z = 0, we get from equation (3.24)

$$r_0 = \frac{k - B_E}{k + B_E}\ ;\ B_E := \frac{-f_E'(0)}{f_E\ (0)}\ .$$

r_0 is the reflection coefficient and $B_E = B_{E1}$ is calculated recursively using equation (3.22). From equation (3.24) we get the relationship:

$$f_E\ (0) = f_E^e\ (0) \frac{2k}{k + B_E} \tag{A.3.25}$$

Thus, for a known source potential $f_E^e\ (0)$ one can calculate $f_E\ (0)$, and from there $\phi_E(0)$ to determine **E** and **H**.

The sources of the TM mode are due to galvanic coupling. If $J_Z^e\ (0)$ is the vertical current density at the surface, (3.11) yields:

$$J_Z^e\ (0) = k^2 \sigma_1\ f_M(0^+\ ) \tag{A.3.26}$$

In this case f_M has to satisfy:

$$\frac{d^2}{d\ z^2}\ f_M(z) = k^2 f_M \tag{A.3.27}$$

and with $B_M := B_{M1} = -\dfrac{f_M'(0)}{f_M\ (0)}$ we get

$$f_M(0^+\ ) = -\frac{k}{B_M}\ f_M(0^-\ ) \tag{A.3.28}$$

because $\dfrac{d\ f_M}{d\ z}$ is continual at $Z = 0$.

Together with equation (3.26) this yields:

$$f_M(0^-\ ,\ \mathbf{k}\ ,\ \omega) = -\frac{B_M(\ \mathbf{k}\ ,\ \omega)}{\sigma_1\ k^3}\ \hat{J}_Z^e(0,\ \mathbf{k}\ ,\ \omega) \tag{A.3.29}$$

In the following we use cylindric coordinates instead of cartesian coordinates which means: $(x, y, z) \rightarrow (r, \phi, z)$ via $x = r\cos\phi,\ \ y = r\sin\phi,\ \ z = z.$

Now we can use the Hankel transformation to calculate ϕ from f:

$$\begin{aligned} \phi\,(z, r, \omega) &= 2\pi \int_0^\infty f\,(k, z, \omega)\; J_0\,(kr)\; kdk \\ f\,(z, k, \omega) &= \frac{1}{2\pi} \int_0^\infty \phi(r, z, \omega)\; J_0\,(kr)\; rdr \end{aligned} \tag{A.3.30}$$

where $r^2 = x^2 + y^2$

For the LOTEM method we have to consider a horizontal electric dipole (HED) which is described by a current I along the x-axis at $\mathbf{r} = 0$ and the electric dipole moment D.

First we want to calculate the external potential $\bar{\phi}^e_E$ for the TE mode (the "-" indicates that the potential is located in the frequency domain). To do this, we use Biot–Savart's law to calculate H_z, and using equation (3.16) we get

$$\bar{H}^e_z\;(r, \omega) = -\frac{D\,y}{4\pi\,R^3} = \frac{1}{\mu_0}\,\frac{\delta^2}{\delta\,z^2}\;\bar{\phi}^e_E\;(\,\mathbf{r}\,,\,\omega)\;. \tag{A.3.31}$$

where $R^2 = r^2 + z^2$

This yields:

$$\frac{\delta^2}{\delta\,z^2}\;\bar{\phi}^e_E\;(\,\mathbf{r}\,,\,\omega) = -\frac{\mu_0\,D\,(\omega)}{4\pi}\;\frac{\delta}{\delta\,y}\left(\frac{1}{R}\right)$$

and, using $\dfrac{1}{R} = \displaystyle\int_0^\infty e^{-k(z)}\;J_1\;(kr)\;dk$, we get

$$\begin{aligned} \bar{\phi}^e_E\,(\,\mathbf{r}\,,\,\omega) &= \frac{\mu_0\,D\,(\omega)}{4\pi} \int_0^\infty e^{-k(z)}\;J_1\;(kr)\;\frac{dk}{k}\;\sin\phi \\ &= \frac{\mu_0\,D\,(\omega)\,y}{4\pi\;(R + |\,z|\;)} \end{aligned} \tag{A.3.32}$$

This results in the total TE potential (using equation (3.24)):

$$\tilde{\phi}_E\;(\,\mathbf{r}\,,\,\omega) = \frac{\mu_0\,D}{4\pi}\left(\frac{r}{R + |z|} - \int_0^\infty B_M\,(k)\;e^{-k(z)}\;J_1\;(kr)\;\frac{dk}{k}\right)\sin\phi \tag{A.3.33}$$

For the TM mode we use the vertical current density at $z = 0$ in the wavenumber domain:

$$\widetilde{J}_z^{e}\ (0,\ \mathbf{k}, \omega) = \frac{d\ i\ k\ x}{4\pi^2}$$

and, using equation (3.29), we get for the total ϕ_M:

$$\widetilde{\phi}_M\ (r\ ,\ \omega) = \frac{-\ D(\omega)}{4\pi} \int_0^{\infty} B_M(k)\ e^{-k|z|}\ J_1\ (kr)\ \frac{dk}{k}\ \cos\phi \qquad (A.3.34)$$

To transform ϕ_E and ϕ_M to time domain, we write the step function

$$D\ (t) = \begin{cases} 0 & t < 0 \\ D_0 & t > 0 \end{cases} \quad \text{as}$$

$$D\ (t) = \frac{D_0}{2\pi\ i} \int_{-\infty}^{\infty} \frac{e^{i\omega t}}{\omega}\ d\ \omega$$

Finally, we calculate **E** and **H** in time domain from ϕ_E and ϕ_M using equation (3.16). The final results for E_x, E_y and d/dt H_z are:

$$E_x^U\ (\mathbf{r},\ t) = \frac{-\ 1}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{i\omega t}}{\omega}\ \frac{-i\ \omega\ \mu_0 D_0}{4\pi} \int_0^{\infty} \left\{ \left(\frac{2\ B_M(\kappa,\omega)\ -\ B_M(\kappa,0)\ -\ \kappa}{\kappa_1^2} \right.\right.$$
$$\left. -\ \frac{2}{B_E\ (\kappa,\omega)\ +\ \kappa} \right) \left(\left(\kappa\ J_0(\kappa, r) - \frac{2}{r}\ J_1\ (\kappa, r) \right) \cos^2\phi + \frac{1}{r}\ J_1\ (\kappa, r) \right)$$
$$\left. -\ \frac{B_E\ (\kappa,\omega)\ -\ \kappa}{B_E\ (\kappa,\omega)\ +\ \kappa}\ J_0(\kappa, r) \right\}\ d\kappa\ d\omega\ -\ \frac{\rho_1\ D_0}{4\pi\ r^3}\ (2\ -\ 3\ \sin^2\phi) \qquad (A.3.35)$$

$$E_y^U\ (\mathbf{r},\ t) = \frac{-\ 2}{2\pi\ i} \int_{-\infty}^{\infty} \frac{e^{i\omega t}}{\omega}\ \frac{-i\ \omega\ \mu_0 D_0\ \cos\phi\ \sin\phi}{4\pi} \int_0^{\infty} \left(\frac{2\ B_M(\kappa,\omega)\ -\ B_M(\kappa,0)\ -\ \kappa}{\kappa_1^2} \right. -$$
$$\left. \frac{2}{B_E\ (\kappa,\omega)\ +\ \kappa} \right) \left(\kappa\ J_0(\kappa, r) - \frac{2}{r}\ J_1\ (\kappa, r) \right) d\kappa\ d\omega\ -\ \frac{3\ \rho_1\ D_0 \cos\phi\ \sin\phi}{4\pi\ r^3}$$

$$U_z^U(\mathbf{r},\ t) = \frac{-\ 1}{2\pi} \int_{-\infty}^{\infty} \mu_0\ A\ e^{i\omega t}\ \frac{D_0\ \cos\phi}{4\pi} \int_0^{\infty} \frac{B_E\ (\kappa,\omega)\ -\ \kappa}{B_E\ (\kappa,\omega)\ +\ \kappa}\ \kappa\ J_1\ (\kappa, r)\ d\kappa\ d\omega \qquad (A.3.36)$$

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